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A POLYNOMIAL INVARIANT OF POLAR LINKS

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Abstract. Knots were generalized to virtual knots by L. H. Kauffman. We introduce polar knots and polar links, another generalization of knots, which are motivated by links in the complement of the z-axis in $\mathbb{R}^3$. There are no knots and no virtual knots with crossing number 1 but we show that there are two polar knots with crossing number 1. We introduce a bracket polynomial of polar links and get an invariant of oriented polar links.

1. Introduction

A *knot* is an embedding of the unit circle S^1 to the 3-dimensional Euclidean space $\mathbb{R}^3$. An n-*component link*, or simply a *link*, is an embedding of n copies of the unit circles to the 3-dimensional Euclidean space $\mathbb{R}^3$. In particular, we see that a 1-component link is a knot. A link can be represented as a diagram in the 2-dimensional Euclidean space $\mathbb{R}^2$ by changing double points of the projection of the link with crossings. A local part of a link diagram as shown in Figure 1 is called a *crossing* of the diagram. For example the link diagram in Figure 2 has 5 crossings.

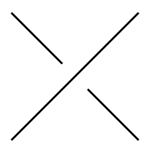

Figure 1. A Crossing.

If a link L is ambient isotopic to another link L' then there is a sequence of moves as shown in Figure 3 from a diagram of L to a diagram of L' ([10]). The moves are called *Reidemeister moves.* Two link diagrams are defined to be *equivalent* if they are related by a finite sequence of Reidemeister moves. L. H. Kauffman introduced virtual knots and virtual links, which generalize knots and links ([9]). Virtual links are motivated by Gauss diagrams under abstractly defined Reidemeister moves for the Gauss diagrams and by links in a thickened surface of higher genus.

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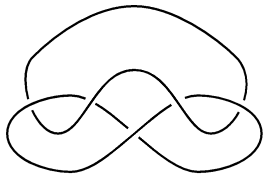

FIGURE 2.

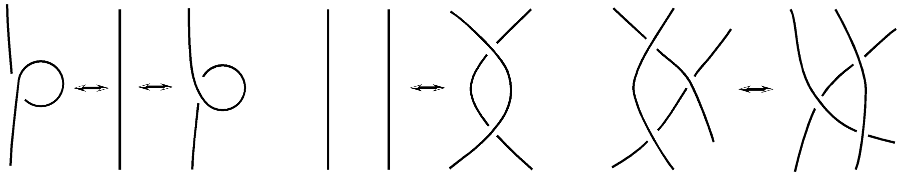

FIGURE 3. Reidemeister moves.

A virtual link can be represented as a virtual link diagram. A virtual link diagram is a link diagram allowed to have virtual crossings. A virtual crossing is denoted by an encircled crossing as shown in Figure 4. For example the virtual link diagram in Figure 5 has 4 virtual crossings.

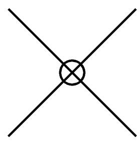

FIGURE 4. A virtual crossing.

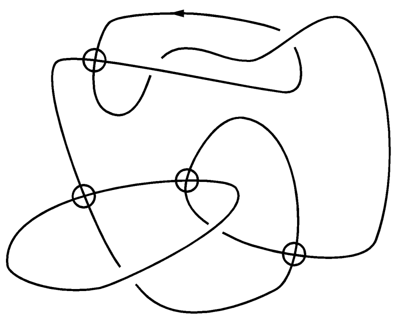

FIGURE 5. A virtual link.

The moves of diagrams as shown in Figure 6 are called *virtual moves.* A sequence of Reidemeister moves and virtual moves is called a *virtual isotopy.* A *virtual knot* is defined to be the virtual isotopy class of a virtual knot diagram. A virtual knot can be represented as a Gauss diagram and vice versa ([1, 9]). If two knot diagrams are virtually isotopic then they are equivalent ([1, 9].

To see whether two virtual knot diagrams are virtually isotopic or not, we use invariants of virtual knots ([1, 5, 9]). In [4]) the author introduced a polynomial of a virtual knot diagram and showed that it gives information about the number of Reidemeister moves of virtual knot diagrams.

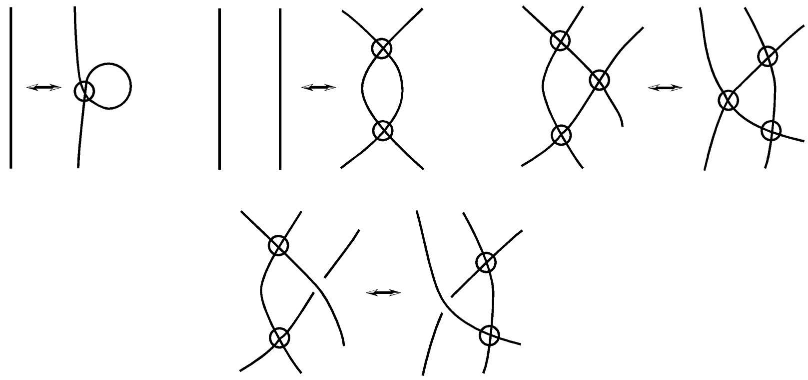

FIGURE 6. Virtual moves.

We will define a polar link which is embedded in $\mathbb{R}^3 - Z$, where Z is the z-axis and equivalence of polar links. We introduce a bracket polynomial and show that polar links can be distinguished by a normalized bracket polynomial.

In Section 2, polar link diagrams and their equivalence are introduced. The equivalence class of a polar link diagram is a polar link. We define the winding number of a polar knot diagram which is a basic invariant of polar knots. Section 3 introduces a bracket polynomial of a polar link diagram. By normalizing the bracket polynomial, we get an invariant of oriented polar links.

2. Polar links and the Winding Number

Consider a link H in $\mathbb{R}^3 - Z$, where Z is the z-axis in $\mathbb{R}^3$. Let P be a plane, which is parallel to the xy-plane, under the link H. Let $P(H)$ be the projection of H and the z-axis Z onto P. Note that the projection of Z onto P is denoted by a dot in $P(H)$. If we give the information of over/under strands for double points of $P(L)$ then it is called a *polar link diagram* or a *polar diagram* of L. See Figure 7 for a polar link diagram. Then a polar link diagram can be represented as link diagram with a dot.

Two polar links diagrams are said to be *equivalent* if they can be related by a finite sequence of Reidemeister moves. The equivalence class of a polar link diagram under the equivalence relation is called a *polar link*. In particular if a polar link diagram has 1 component then it is called a *polar knot diagram* and its equivalence class under the equivalence relation is called a *polar knot*. The number of crossings in a polar diagram is called its *crossing number*. For a polar link L, there are many polar link diagrams D

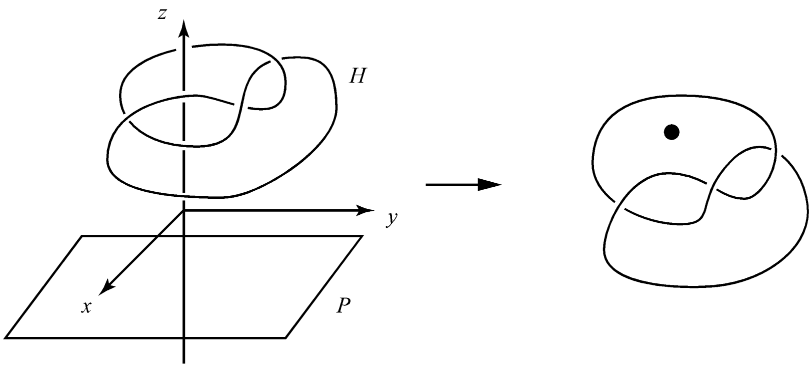

FIGURE 7. A polar link and its polar link diagram.

which represent L. The minimal number of the crossings in D representing L is called the *crossing number* of L.

A polar link diagram L is said to be *oriented* if each component of L has a direction which is indicated by arrows.

(sign) We define the *sign* of a crossing of an oriented polar link diagram as shown in Figure 8. The *writhe* $w(K)$ of a knot diagram K is defined to be the sum of signs of all crossings of K.

FIGURE 8. The sign of a crossing.

A function from the set of all polar knot (link) diagrams to a set S is called *polar knot (link) invariant* if it takes the same value for equivalent polar knot (link) diagrams. If two polar knot diagrams take different values for a polar knot invariant then the two polar knot diagrams are not equivalent.

For an oriented polar knot diagram K, let P_K be the oriented closed curve with a dot in the plane obtained from K by changing crossings to double points. Let W^+ (W^-) be the number of times that the closed curve of P_K travels counterclockwise (clockwise) around the dot. The *winding number* of K is defined to be the difference $W^+ - W^-$. For example the winding number of the polar knot diagram K in Figure 9 is 2.

Since the winding number is invariant under the Reidemeister moves, we get the following

Lemma 2.1. *The winding number is a polar knot invariant.*

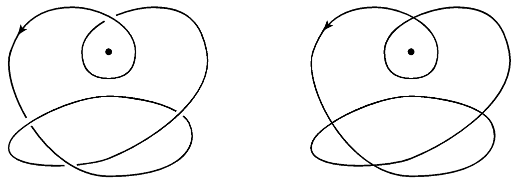

FIGURE 9.

Example 2.2. Let K_1 and K_2 be the polar knot diagrams as shown in Figure 21. Since the winding numbers of K_1 and K_2 are 2 and 0 respectively, K_1 and K_2 are not equivalent.

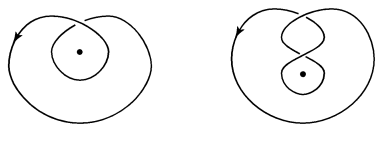

FIGURE 10.

3. A POLYNOMIAL INVARIANT OF POLAR LINKS

After the discovery of a new polynomial invariant by V. F. R. Jones ([6]), many new polynomial invariants were introduced ([2, 3, 8]). L. H. Kauffman introduced a state model in [7] for the Jones polynomial and extended it to virtual knots in [9]. We introduce a polynomial invariant of polar links by using a state sum model, motivated by the Kauffman bracket polynomial. The bracket polynomial was introduced by L. H. Kauffman ([7]) and we will modify it to get an invariant of polar links.

Definition 3.1. We define the bracket polynomial of a polar link diagram by using the following axioms.

(1) Axiom (1) $\langle L\rangle = A\langle L_0\rangle + A^{-1}\langle L_\infty\rangle$, where L, L_O and L_∞ are polar link diagrams which differ as shown in Figure 11 and A is an indeterminate.
(2) Axiom (2) $\langle K \cup O\rangle = (-A^2 - A^{-2})\langle K\rangle$, where $K \cup O$ is a disjoint union of a polar link diagram K and a circle component O, which does not contain the dot.
(3) Axiom (3) $\langle O_n\rangle = C^n$, where O_n is a diagram with n circle components as shown in Figure 12, and C is an indeterminate.

Example 3.2. Let K be a polar knot diagram as shown in Figure 13. Then we can see that $\langle K\rangle = -A^4 - 1 + C^2 - A^{-4}C^2$ as calculated in Figure 14.

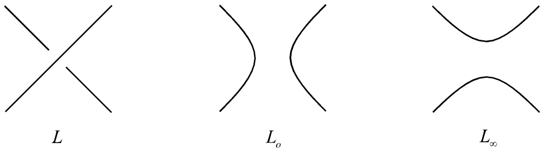

FIGURE 11.

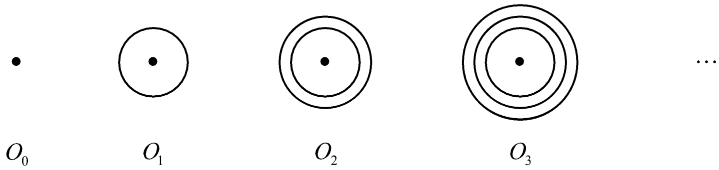

FIGURE 12.

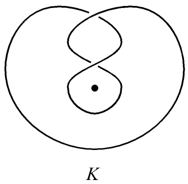

FIGURE 13.

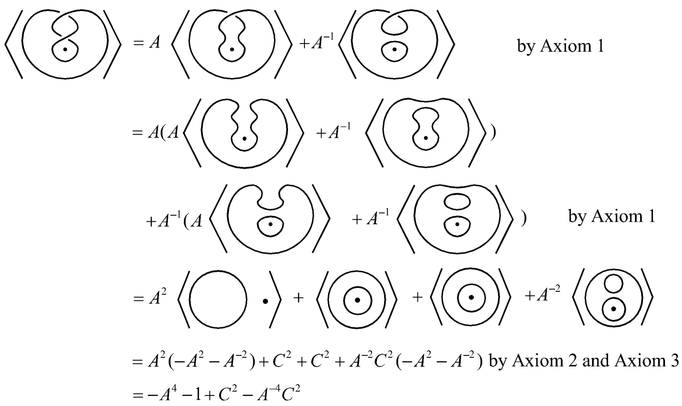

FIGURE 14.

In calculation of the bracket polynomial, we will often denote $\langle K\rangle$ by K. Now we will show that the bracket polynomial is invariant under the second Reidemeister move and the third Reidemeister move. Lemma 3.3, Lemma3.4 and Lemma 3.5 are proved for link diagrams by L. H. Kauffman ([7]), we show the lemmas hold for polar link diagrams.

Lemma 3.3. *Let L_1 and L_2 be polar diagrams which are related by a second Reidemeister move as shown in Figure 15. Then $\langle L_1 \rangle = \langle L_2 \rangle$.*

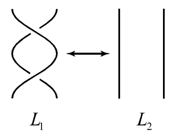

FIGURE 15.

Proof. See Figure 16.

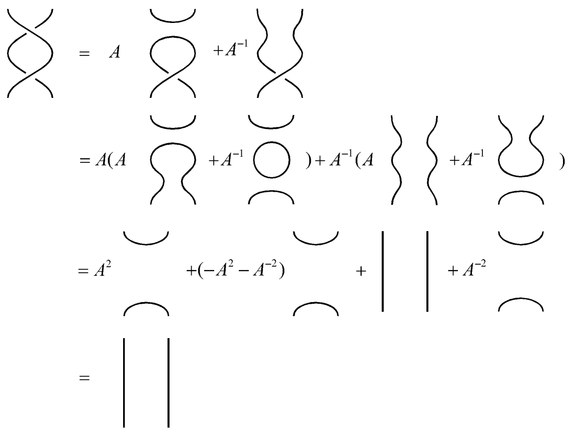

FIGURE 16. The bracket polynomial is invariant under the second Reidemeister move.

□

Lemma 3.4. *Let L_1 and L_2 be polar diagrams which are related by a third Reidemeister move as shown in Figure 17. Then $\langle L_1 \rangle = \langle L_2 \rangle$.*

Proof. See Figure 18, where the second and the third equalities come from Lemma 3.3 □

The bracket polynomial is invariant under the second Reidemeister move and the third Reidemeister move, but it is not under the first Reidemeister move.

Lemma 3.5. *Let L_1, L_2 and L_3 be polar diagrams which are related by first Reidemeister moves as shown in Figure 19. Then $\langle L_1 \rangle = (-A^3)\langle L_2 \rangle$ and $\langle L_3 \rangle = (-A^{-3})\langle L_2 \rangle$.*

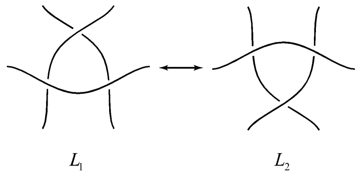

FIGURE 17.

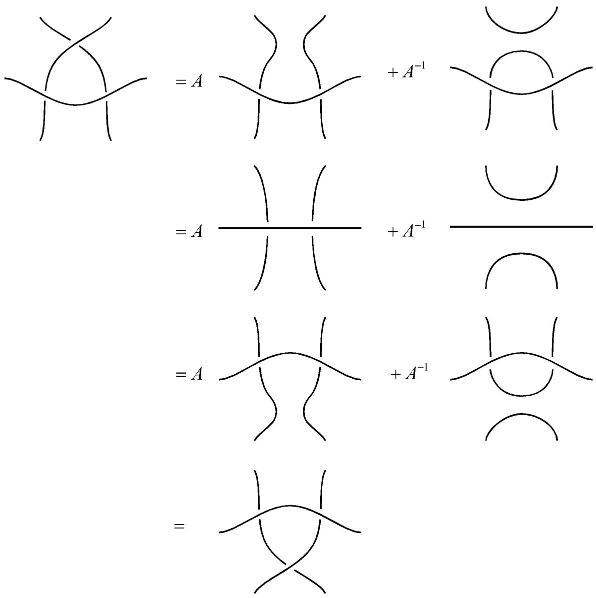

FIGURE 18.

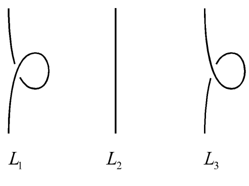

FIGURE 19.

Proof. Figure 20 shows that $\langle L_1 \rangle = (-A^3)\langle L_2 \rangle$. Similarly, we see that $\langle L_3 \rangle = (-A^{-3})\langle L_2 \rangle$. □

L. H. Kauffman introduced an invariant of links by normalizing the bracket polynomial by using the writhe ([7]). Similarly, we will get an invariant of polar links by normalizing the bracket polynomial of polar links. While the Kauffman polynomial has

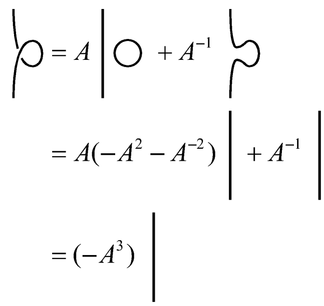

FIGURE 20.

one indeterminate A, our polynomial has two indeterminates A and C. If the polar link diagrams L_1, L_2 and L_3 in Figure 19 are oriented then the writhes of the diagrams differ as the equation $w(L_1) - 1 = w(L_2) = w(L_3) + 1$. Since the writhe of a polar link diagrams is invariant under the second Reidemeister move and the third Reidemeister move, we may guess that an invariant of polar links can be obtained by combining the writhe and the bracket polynomial. For an oriented link diagram L, we define

$$X_L(A, C) = (-A^3)^{-w(L)}\langle L\rangle,$$

where $\langle L\rangle$ is the bracket polynomial of the diagram obtained from L by removing its orientation.

Theorem 3.6. *$X_L(A, C)$ is an invariant of polar links.*

Proof. Since the writhe and the bracket polynomial are invariant under the second Reidemeister move and the third Reidemeister move, it is enough to show that the polynomial $X_L(A, C)$ is invariant under the first Reidemeister move. Let L_1, L_2 and L_3 be oriented polar link diagrams which differs locally as shown in Figure 19. By Lemma 3.5, $\langle L_1\rangle = (-A^3)\langle L_2\rangle$. Since $w(L_1) = w(L_2) + 1$, we see that

$$\begin{aligned} X_{L_1}(A, C) &= (-A^3)^{-w(L_1)}\langle L_1\rangle \\ &= (-A^3)^{-w(L_2)-1}(-A^3)\langle L_2\rangle \\ &= (-A^3)^{-w(L_2)}\langle L_2\rangle \\ &= X_{L_2}(A, C). \end{aligned}$$

Similarly, we see that $X_{L_3}(A, C) = X_{L_2}(A, C)$. □

For a polar knot diagram D, we can give orientations on D in two ways. Let K_1 and K_2 are the two oriented knot diagrams obtained from D. Then the writhes of K_1 and K_2 are the same and $X_{K_1}(A, C) = X_{K_2}(A, C)$. We define $X_D(A, C) = X_{K_1}(A, C)$.

Example 3.7. The polar knots K_1, K_2, K_3 and K_4 in Figure 21 are all polar knots whose crossing numbers are less than 3. We can show that the knots are inequivalent by comparing the X-polynomials. By direct calculation we can see that $X_{K_1}(A, C) = A^{-6} + A^{-2} - A^{-2}C^2$, $X_{K_2}(A, C) = A^2 + A^6 - A^2C^2$, $X_{K_3}(A, C) = -A^4 - A^6 - A^8 -$

$A^{10} + (A^4 + A^6)C^2$, and $X_{K_4}(A, C) = -A^{-10} - A^{-8} - A^{-6} - A^{-4} + (A^{-6} + A^{-4})C^2$. Therefore the four polar knots K_1, K_2, K_3 and K_4 are distinct.

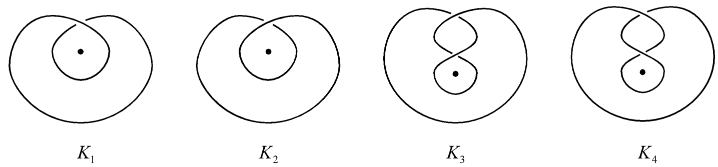

FIGURE 21. All polar knots with crossing number < 3.

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